

Tunnelling of scalar and Dirac particles from squashed charged rotating Kaluza-Klein black holes

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Abstract

Thermal radiation of scalar particles and Dirac fermions from squashed charged rotating five-dimensional black holes is considered. To obtain temperature of the black holes we use the tunnelling method. In case of scalar particles we make use of the Hamilton-Jacobi equation. To consider tunnelling of fermions the Dirac equation was investigated. The examination shows that radial parts of the action for scalar particles and fermions in quasi-classical limit in the vicinity of horizon are almost the same and as a consequence it gives rise to the identical expressions for the temperature in both cases.

1 Introduction

Hawking radiation has been investigated since the early 70-ies of the last century [1, 2]. To find temperature of black holes different methods were used. Semiclassical tunnelling method proposed by Kraus and Wilczek [3, 4] and developed in works of Parikh and Wilczek [5] has been attracted a lot of interest recently. We remark that nowadays the tunnelling method is comprised of two different approaches namely the approach proposed by Parikh and Wilczek and it is called the null-geodesic method and the second one that is known as the Hamilton-Jacobi method [6, 7]. It is worth noting that null-geodesic method is truly semiclassical because it is based on the equation of motion of a classical particle that moves along a null-geodesic. The Hamilton-Jacobi method is based on equations that we deal with in quantum theory such as the Klein-Gordon equation for scalar particles and the Dirac equation for fermions with spin $s = 1/2$. For particles of higher spins Rarita-Schwinger or Proca equations can be used. So here one departs from quantum equation of motion and then considers quasi-classical limit. Despite the different starting points two methods give rise to the same temperatures if the black hole's metrics is the same.

These approaches were applied to numerous examples of black holes and obtained results were in agreement with expressions for temperature calculated by other methods. Among them we distinguish Kerr and Kerr-Newman black holes' space-time [8, 9, 10, 11, 12], Taub-NUT space-time [13] Gödel space-time [14], BTZ black holes [15, 16], dynamical black holes [17], black holes in Hořava-Lifshitz gravity [18, 19], accelerating and rotating black hole [20, 21],

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rotating black strings [22] and many others. The review of tunnelling method was considered in paper [23] where further references on that subject are given.

Multidimensional black holes have been attracted great attention in recent years [24]. There are great expectations that multidimensional micro black holes can appear in high energy particles collisions which are based on the fact that extra spacelike dimensions can lower the Planck scale up to the *TeV*-energy region [25, 26]. Squashed Kaluza-Klein black holes represent one of the most interesting solutions among higher dimensional black holes. The first five-dimensional Kaluza-Klein black hole's solution was found by Dobiasch and Maison [27] and that metrics was studied in [28]. The solution was generalized to the five-dimensional Einstein-Maxwell theory [29]. To find the solution so called squashing transformation was used [29]. The squashing transformation was applied to construct rotating black hole solution [30], charged rotating black hole [31], charged rotating black hole in Gödel universe [32, 33]. A review of Kaluza-Klein black holes' solutions can be found in [34].

Different aspects of Kaluza-Klein black holes were also considered, namely thermodynamics [35, 36, 37], Hawking radiation and tunnelling method [38, 39, 40, 41, 42, 43, 44], quasinormal modes and stabilities [45, 46, 47, 48, 49], geodetic precession [50], gravitational lensing [51].

In our work we consider scalar particles and fermion tunnelling for charged rotating Kaluza-Klein black holes. The paper is organized as follows. In the second section we investigate tunnelling of scalar particles. In the third section the tunnelling of Dirac fermions is examined. The forth section contains some conclusions.

2 Tunnelling of scalar particles

Let us consider tunnelling of scalar particles. The Klein-Gordon equation for charged massive particles can be written in the form:

$$\frac{1}{\sqrt{-g}} (\partial_\mu - ieA_\mu) \sqrt{-g} g^{\mu\nu} (\partial_\nu - ieA_\nu) \Psi - \tilde{m}^2 \Psi = 0. \quad (1)$$

The tunnelling process is supposed to be quasi-classical. To consider it we assume that quasi-classical wave function takes form:

$$\Psi = C \exp \left\{ \frac{i}{\hbar} I_\uparrow \right\} \quad (2)$$

Where $I_\uparrow = I_\uparrow(t, r, \theta, \phi, \psi)$ denotes a quasi-classical action of emitted particles. In the first order approximation the Klein-Gordon equation (1) leads to a Hamilton-Jacobi equation for the relativistic particle:

$$g^{\mu\nu} (\partial_\mu I_\uparrow \partial_\nu I_\uparrow + e^2 A_\mu A_\nu - 2e A_\mu \partial_\nu I_\uparrow) + \tilde{m}^2 = 0. \quad (3)$$

We will examine tunnelling of particles through the horizon of squashed charged rotating Kaluza-Klein black hole whose metric and gauge potential were obtained in [31]:

$$ds^2 = -\frac{w(r)}{h(r)} dt^2 + \frac{k^2(r)}{w(r)} dr^2 + \frac{r^2}{4} [k(r)(\sigma_1^2 + \sigma_2^2) + h(r)(f(r)dt + \sigma_3)^2] \quad (4)$$

where functions $w(r)$, $h(r)$, $k(r)$ and $f(r)$ are defined as follows

$$w(r) = \frac{(r^2 + q)^2 - 2(m + q)(r^2 - a^2)}{r^4} = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^4} \quad (5)$$

$$h(r) = 1 - \frac{a^2 q^2}{r^6} + \frac{2a^2(m + q)}{r^4} \quad (6)$$

$$k(r) = \frac{(r_\infty^2 + q)^2 - 2(m + q)(r_\infty^2 - a^2)}{(r_\infty^2 - r^2)^2} = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{(r_\infty^2 - r^2)^2} \quad (7)$$

$$f(r) = -\frac{2a}{r^2 h(r)} \left(\frac{2m + q}{r^2} - \frac{q^2}{r^4} \right) \quad (8)$$

Gauge potential is defined by the relation:

$$A = \frac{\sqrt{3}q}{2r^2} \left(dt - \frac{a}{2}\sigma_3 \right) \quad (9)$$

and left-invariant 1-forms on S^3 are given by

$$\sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi \quad (10)$$

$$\sigma_2 = -\sin \psi d\theta + \cos \psi \sin \theta d\phi \quad (11)$$

$$\sigma_3 = d\psi + \cos \theta d\phi \quad (12)$$

The coordinates $(t, r, \theta, \phi, \psi)$ run the ranges of $-\infty < t < +\infty$, $0 < r < r_\infty$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$, $0 \leq \psi < 4\pi$ respectively. The case $k(r) = 1$ or equivalently $r_\infty \rightarrow \infty$ leads to the Cvetič solution [52, 53]. The parameters a , m , q and r_∞ fulfil the inequalities:

$$m > 0, \quad (13)$$

$$q^2 + 2(m + q)a^2 > 0, \quad (14)$$

$$(r_\infty^2 + q)^2 - 2(m + q)(r_\infty^2 - a^2) > 0, \quad (15)$$

$$(m + q)(m - q - 2a^2) > 0, \quad (16)$$

$$m + q > 0. \quad (17)$$

It was pointed out by the authors [31] that inequalities (13)-(16) are necessary for the existence of two horizons and the last inequality (17) provides the absence of closed timelike curves outside the outer horizon. The horizons are defined by the equation $w(r) = 0$ and it leads to: $r_+^2 = m + \sqrt{(m + q)(m - q - 2a^2)}$ and $r_-^2 = m - \sqrt{(m + q)(m - q - 2a^2)}$. The metric (4) also diverges at $r = r_\infty$ but it is an apparent singularity corresponding to the spatial infinity. To prove it one should make a coordinate transformation

$$\rho = \rho_0 \frac{r^2}{r_\infty^2 - r^2} \quad (18)$$

where ρ_0 is defined by

$$\rho_0^2 = \frac{(r_\infty^2 + q)^2 - 2(m + q)(r_\infty^2 - a^2)}{4r_\infty^2} = \frac{(r_\infty^2 - r_+^2)(r_\infty^2 - r_-^2)}{4r_\infty^2} \quad (19)$$

It is clear when $r \rightarrow r_\infty$ then $\rho \rightarrow \infty$. It was also shown [31] that asymptotic time (time for a distant observer) differs from the coordinate one and takes form:

$$\tilde{t} = \frac{2r_\infty^2 \rho_0}{\sqrt{r_\infty^6 - a^2(q^2 - 2(m + q)r_\infty^2)}} t = \beta t \quad (20)$$

Now we consider the Hamilton-Jacobi equation (3) in case of the black hole's metric (4). The functions $w(r)$, $k(r)$, $h(r)$ and $f(r)$ depend only on the radial variable r and it gives rise to the conclusion that the angular variables can be separated from the radial one (or at least some

of them). Using this fact we suppose that the angular variables ϕ and ψ can be separated. So the action for emitted particle I_{\uparrow} might be written in the form

$$I_{\uparrow} = -E\tilde{t} + W(r, \theta) + J\phi + L\psi \quad (21)$$

where J and L are constants and \tilde{t} denotes time for a distant observer which is defined by the relation (20).

As a result the Hamilton-Jacobi equation can be represented in the form:

$$\begin{aligned} & \frac{w(r)}{k^2(r)} W_r^2 + \frac{4W_\theta^2}{r^2 k(r)} - \frac{h(r)}{w(r)} \left(\beta E + f(r)L + \frac{\sqrt{3}qe}{2r^2} \left(1 + \frac{af(r)}{2} \right) \right)^2 \\ & + \frac{1}{r^2 h(r)} \left(2L + \frac{\sqrt{3}qea}{2r^2} \right)^2 + \frac{4}{r^2 k(r)} \left(\frac{J}{\sin \theta} - L \cot \theta \right)^2 + \tilde{m}^2 = 0 \end{aligned} \quad (22)$$

One can see that variables r and θ can be separated in the same way. So the function $W(r, \theta)$ can be written as follows:

$$W(r, \theta) = R(r) + \Theta(\theta) \quad (23)$$

Then for radial part we obtain:

$$\begin{aligned} R'(r) = & \frac{\beta k(r) \sqrt{h(r)}}{w(r)} \left[\left(E + \frac{f(r)}{\beta} L + \frac{\sqrt{3}qe}{2\beta r^2} \left(1 + \frac{af(r)}{2} \right) \right)^2 \right. \\ & \left. - \frac{w(r)}{\beta^2 h(r)} \left(\frac{1}{r^2 h(r)} \left(2L + \frac{\sqrt{3}qea}{2r^2} \right)^2 + \frac{4D}{r^2 k(r)} + \tilde{m}^2 \right) \right]^{1/2} \end{aligned} \quad (24)$$

where D is a constant that appeared after separation of variables.

In the vicinity of the horizon point $w(r_+) = 0$ one should use the decomposition $w(r) = w'(r_+)(r - r_+) = 2(r_+^2 - r_-^2)(r - r_+)/r_+^3$. Then the latter equation can be written in the form:

$$R'(r) = \frac{\beta r_\infty^2 - r_-^2}{2 r_\infty^2 - r_+^2} \frac{\sqrt{r_+^6 - a^2 q^2 - 2a^2(m+q)r_+^2}}{(r_+^2 - r_-^2)(r - r_+)} \left(E + \frac{f(r_+)L}{\beta} + \frac{\sqrt{3}qe}{2\beta r_+^2} \left(1 + \frac{af(r_+)}{2} \right) \right) \quad (25)$$

Integrating this expression in the vicinity of the horizon point we obtain:

$$R(r) = \frac{\beta r_\infty^2 - r_-^2}{2 r_\infty^2 - r_+^2} \frac{\sqrt{r_+^6 - a^2 q^2 - 2a^2(m+q)r_+^2}}{(r_+^2 - r_-^2)} \left(E + \frac{f(r_+)L}{\beta} + \frac{\sqrt{3}qe}{2\beta r_+^2} \left(1 + \frac{af(r_+)}{2} \right) \right) \int_{r_+ - \varepsilon}^{r_+ + \varepsilon} \frac{dr}{(r - r_+)} \quad (26)$$

We consider the tunnelling process and it means that the action for the emitted particles I_{\uparrow} (21) takes complex values. This fact immediately follows from the form of the integral for the radial part of the action (26). The function we integrate in (26) has a pole at the point r_+ . Then we should integrate around the pole and as a result the complex values appear. To obtain temperature of the black hole we take into account only the radial part of the action (26). So we can write:

$$Im R_{\uparrow} = \pi \frac{\beta r_\infty^2 - r_-^2}{2 r_\infty^2 - r_+^2} \frac{\sqrt{r_+^6 - a^2 q^2 - 2a^2(m+q)r_+^2}}{(r_+^2 - r_-^2)} \left(E + \frac{f(r_+)L}{\beta} + \frac{\sqrt{3}qe}{2\beta r_+^2} \left(1 + \frac{af(r_+)}{2} \right) \right) \quad (27)$$

It was supposed [6, 23] that probabilities of crossing a black hole's horizon can be defined as:

$$P_{out} \propto \exp\{-2ImR_{\uparrow}\}, \quad P_{int} \propto \exp\{-2ImR_{\downarrow}\} \quad (28)$$

The ratio for these two probabilities leads to the Boltzmann factor $\exp\{-E/T\}$ which shows that radiation is thermal [54]:

$$\Gamma = \frac{P_{out}}{P_{in}} = \frac{\exp\{-2ImR_{\uparrow}\}}{\exp\{-2ImR_{\downarrow}\}} = \exp\left\{-\frac{E}{T}\right\}. \quad (29)$$

The imaginary part of the radial part of the action for ingoing particles can be calculated in the same way as it was carried out for the outgoing case. The resulting imaginary part will take only the opposite sign in comparison to the expression (27), so $ImR_{\downarrow} = -ImR_{\uparrow}$. Substituting the expression (26) and taking into consideration mentioned above remark we arrive at:

$$\Gamma = \exp\left\{-2\pi\beta \frac{r_{\infty}^2 - r_+^2}{r_{\infty}^2 - r_+^2} \frac{\sqrt{r_+^6 - a^2q^2 - 2a^2(m+q)r_+^2}}{(r_+^2 - r_-^2)} \left(E + \frac{f(r_+)L}{\beta} + \frac{\sqrt{3}qe}{2\beta r_+^2} \left(1 + \frac{af(r_+)}{2}\right)\right)\right\} \quad (30)$$

Having supposed that emission of particles is thermal we obtain the temperature of the black hole:

$$T = \frac{1}{2\pi} \frac{r_+^2 - r_-^2}{r_{\infty}(r_{\infty}^2 - r_-^2)} \sqrt{\frac{r_{\infty}^2 - r_+^2}{r_{\infty}^2 - r_-^2}} \sqrt{\frac{r_+^6 - a^2(q^2 - 2(m+q)r_{\infty}^2)}{r_+^6 - a^2(q^2 - 2(m+q)r_+^2)}} \quad (31)$$

The result we obtain is in agreement with the expression for the surface gravity that was found in [31].

Now we consider a squashed black hole in Gödel universe. The metric of the black hole takes the following form [32]:

$$ds^2 = -k(r)dt^2 - 2g(r)\sigma_3 dt + h(r)\sigma_3^2 + \frac{\chi^2(r)}{V(r)}dr^2 + \frac{r^2}{4} [\chi(r)(\sigma_1^2 + \sigma_2^2)] \quad (32)$$

where

$$k(r) = 1 - \frac{2m}{r^2} + \frac{q^2}{r^4} \quad (33)$$

$$g(r) = jr^2 + 3jq + \frac{(2m-q)a}{2r^2} - \frac{q^2a}{2r^4} \quad (34)$$

$$h(r) = \frac{r^2}{4} - j^2r^2(r^2 + 2m + 6q) + 3jq a + \frac{(m-q)a^2}{2r^2} - \frac{q^2a^2}{4r^4}; \quad (35)$$

$$V(r) = 1 + \frac{8j(m+q)(a + 2j(m+2q)) - 2m}{r^2} + \frac{2(m-q)a^2 + q^2[1 - 16ja - 8j^2(m+3q)]}{r^4}; \quad (36)$$

$$\chi(r) = \frac{c^2 + 2c(m - 4j(m+q)[a + 2j(m+2q)]) + q^2 + 2a^2(m-q) - 8q^2j[2a + j(m+3q)]}{(r^2 + c)^2} \quad (37)$$

We note that here we keep notations given in the paper [32]. The squashing functions here is $\chi(r)$. It was shown that in case when Gödel parameter $j = 0$ we arrive at the previously considered metric (4). We also note that the squashing parameter c has to be chosen negative $c = -r_0^2$ (the parameter r_0 corresponds to the parameter r_{∞} of previously considered metric). The electromagnetic one-form potential can be written as follows:

$$A = \left(\frac{\sqrt{3}q}{2r^2} - \Phi\right) dt + \frac{\sqrt{3}}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2}\right) \sigma_3 \quad (38)$$

and here Φ is a constant which one can find from the requirement that electromagnetic potential should be regular at the horizon. Horizon points for the black hole given by the metric (32) can be found as roots of equation $V(r) = 0$ [32]:

$$r_{\pm}^2 = m - 4j(m+q)(a+2j(m+2q)) \pm \sqrt{\delta}, \quad (39)$$

$$\delta = (m-q-8j^2(m+q)^2)[m+q-2a^2-8ja(m+2q)-8j^2(m+2q)^2] \quad (40)$$

We note that in the limit $j = 0$ we recover the previously considered black hole's solution (4).

It should be noticed that black hole solutions in Gödel universe possess close timelike curves (CTC). In our case we will consider situation when $r_- < r_+ < r_0 < r_{CTC}$ so this peculiarity of the solution (32) is not important for us.

Now we examine the Hamilton-Jacobi equation for the squashed Gödel black hole. Similarly as in the previous case components of metric tensor do not depend explicitly on the coordinates t, ϕ, ψ and it allows us to separate those angular and time coordinates. So the action for an emitted particle can be chosen in the form (21). Having accomplished the separation of variables we write the Hamilton-Jacobi equation:

$$\begin{aligned} \frac{V(r)}{\chi^2(r)} W_r^2 + \frac{4}{r^2 \chi(r)} \left(W_\theta^2 + \frac{(J - L \cos \theta)^2}{\sin^2 \theta} \right) + \frac{4}{r^2 V(r)} \left(-h(r) \left[E + e \left(\frac{\sqrt{3}q}{2r^2} - \Phi \right) \right]^2 + \right. \\ \left. 2g(r) \left[\left(E + e \left(\frac{\sqrt{3}q}{2r^2} - \Phi \right) \right) \left(L - \frac{\sqrt{3}e}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right) \right] \right. \\ \left. + k(r) \left(L - \frac{\sqrt{3}e}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right)^2 \right) + \tilde{m}^2 = 0 \quad (41) \end{aligned}$$

We note that here we have used the relation $g^2(r) + k(r)h(r) = r^2 V(r)/4$ which can be verified easily. Similarly to the previous case the radial r and angular θ variables can be separated. So we make use of the relation (23) and write the relation for the derivative of radial part of action I_\uparrow

$$\begin{aligned} R'(r) = \frac{2\chi(r)\sqrt{h(r)}}{rV(r)} \left(\left[E + e \left(\frac{\sqrt{3}q}{2r^2} - \Phi \right) - \frac{g(r)}{h(r)} \left(L - \frac{\sqrt{3}e}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right) \right]^2 - \right. \\ \left. \frac{r^2 V(r)}{h(r)} \left[\frac{1}{h(r)} \left(L - \frac{\sqrt{3}e}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right)^2 + \frac{4D}{r^2 \chi(r)} + \tilde{m}^2 \right] \right)^{1/2} \quad (42) \end{aligned}$$

Integrating the obtained relation around the horizon point r_+ and taking the imaginary part of it we arrive at:

$$Im R_\uparrow = \pi \frac{r_+^2 \chi(r_+) \sqrt{h(r_+)}}{(r_+^2 - r_-^2)} \left[E + e \left(\frac{\sqrt{3}q}{2r_+^2} - \Phi \right) - \frac{g(r_+)}{h(r_+)} \left(L - \frac{\sqrt{3}e}{2} \left(jr_+^2 + 2jq - \frac{aq}{2r_+^2} \right) \right) \right] \quad (43)$$

To find temperature of the black hole we make use of the same procedure as we have done earlier. It should be emphasized that similarly to the previous metric to obtain correct relation for temperature we have to take into account the fact that a distant observer uses asymptotic time but not a coordinate one. So we perform transformation similar to (20) $t \rightarrow t/N_0$, where parameter N_0 takes form:

$$N_0^2 = \frac{r_0^2 V(r_0)}{4h(r_0)}. \quad (44)$$

This transformation leads to the replacement $E \rightarrow EN_0$ in the right hand side of the relation (43). We note that in case of previously considered metric the situation was the same.

So we write:

$$T = \frac{1}{2\pi} \frac{r_0}{r_+^2} \frac{r_+^2 - r_-^2}{r_0^2 - r_-^2} \sqrt{\frac{r_0^2 - r_+^2}{r_0^2 - r_-^2}} \sqrt{\frac{h(r_0)}{h(r_+)}} \quad (45)$$

In the limit when the Gödel parameter is equal to zero ($j = 0$) the obtained relation (45) can be represented in the form (31).

3 Tunnelling of a charged spin-1/2 particle from squashed Kaluza-Klein black hole

In this section we examine the tunnelling method for Dirac particles. For the first time tunnelling method for fermions was considered by Kerner and Mann [11]. Then it was successfully applied to the vast area of black holes [23]. In case of scalar particles the starting point was the Klein-Gordon equation (1) which leads to the Hamilton-Jacobi equation (3) in the quasi-classical limit. To investigate the tunnelling of fermions the Klein-Gordon equation should be replaced by the Dirac equation. Then similarly to the case of scalar particles quasi-classical limit should be taken. Firstly we consider black hole's metric (4) and then go to a bit more general metric of the squashed black hole in Gödel universe (32). The Dirac equation for an electrically charged particle takes form:

$$i\gamma^\mu \left(D_\mu - \frac{ie}{\hbar} A_\mu \right) \Psi + \frac{\tilde{m}}{\hbar} \Psi = 0 \quad (46)$$

where $D_\mu = \partial_\mu + \Omega_\mu$, $\Omega_\mu = \frac{1}{8} \Gamma_\mu^{\alpha\beta} [\gamma^\beta, \gamma^\alpha]$ and γ^μ matrices fulfil commutation relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \hat{1} \quad (47)$$

Matrices γ^μ can be defined in different manners and in our work we take them in the following form:

$$\begin{aligned} \hat{\gamma}^{\tilde{t}} &= \beta \sqrt{\frac{h(r)}{w(r)}} \hat{\gamma}^0, \quad \hat{\gamma}^r = \frac{\sqrt{w(r)}}{k(r)} \hat{\gamma}^3, \quad \hat{\gamma}^\theta = \frac{2}{r\sqrt{k(r)}} \hat{\gamma}^1, \\ \hat{\gamma}^\phi &= \frac{2}{r\sqrt{k(r)} \sin \theta} \hat{\gamma}^2, \quad \hat{\gamma}^\psi = -f(r) \sqrt{\frac{h(r)}{w(r)}} \hat{\gamma}^0 - \frac{2 \cot \theta}{r\sqrt{k(r)}} \hat{\gamma}^2 + \frac{2}{r\sqrt{h(r)}} \hat{\gamma}^4 \end{aligned} \quad (48)$$

It should be noted that we consider black hole's metric (4). Matrices γ^A take form:

$$\gamma^0 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \gamma^1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} \quad (49)$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad \gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} \quad \gamma^4 = \begin{pmatrix} -I & 0 \\ 0 & -I \end{pmatrix} \quad (50)$$

and σ_i are the Pauli matrices.

So the Dirac equation can be written in the form:

$$\begin{aligned}
& i \left[\beta \sqrt{\frac{h(r)}{w(r)}} \hat{\gamma}^0 \partial_{\bar{t}} + \frac{\sqrt{w(r)}}{k(r)} \hat{\gamma}^3 \partial_r + \frac{2}{r\sqrt{k(r)}} \hat{\gamma}^1 \partial_\theta + \frac{2}{r\sqrt{k(r)} \sin \theta} \hat{\gamma}^2 \partial_\phi \right. \\
& \quad \left. + \left(-f(r) \sqrt{\frac{h(r)}{w(r)}} \hat{\gamma}^0 - \frac{2 \cot \theta}{r\sqrt{k(r)}} \hat{\gamma}^2 + \frac{2}{r\sqrt{h(r)}} \hat{\gamma}^4 \right) \partial_\psi \right] \Psi + \\
& \quad \frac{\sqrt{3}qe}{2\hbar r^2} \left[\sqrt{\frac{h(r)}{w(r)}} \left(1 + \frac{af(r)}{2} \right) \hat{\gamma}^0 - \frac{a}{r\sqrt{h(r)}} \hat{\gamma}^4 \right] \Psi + \frac{\tilde{m}}{\hbar} \Psi = 0
\end{aligned} \tag{51}$$

We also remark that here spin connection terms Ω_μ are omitted because we will consider quasi-classical limit and take the lowest order approximation whereas spin connection gives rise to the terms of the next order.

The wave functions with spin up and down can be chosen in the form:

$$\Psi_\uparrow = \begin{pmatrix} A(t, r, \theta, \phi, \psi) \\ 0 \\ B(t, r, \theta, \phi, \psi) \\ 0 \end{pmatrix} \exp \left(\frac{i}{\hbar} I_\uparrow(t, r, \theta, \phi, \psi) \right); \quad \Psi_\downarrow = \begin{pmatrix} 0 \\ C(t, r, \theta, \phi, \psi) \\ 0 \\ B(t, r, \theta, \phi, \psi) \end{pmatrix} \exp \left(\frac{i}{\hbar} I_\downarrow(t, r, \theta, \phi, \psi) \right), \tag{52}$$

where I_\uparrow and I_\downarrow are the action for the Dirac particles with spin-up (\uparrow) and spin-down (\downarrow) tunnelling through the horizon.

Substituting (52) into the Dirac equation (51) and taking the lowest order terms (the terms proportional to \hbar^{-1}) we arrive at

$$\begin{aligned}
& B \left(\sqrt{\frac{h(r)}{w(r)}} \left[-\beta \partial_{\bar{t}} I_\uparrow + f(r) \partial_\psi I_\uparrow + \frac{\sqrt{3}qe}{2r^2} \left(1 + \frac{af(r)}{2} \right) \right] - \right. \\
& \quad \left. \frac{\sqrt{w(r)}}{k(r)} \partial_r I_\uparrow \right) + A \left(\frac{2}{r\sqrt{h(r)}} \partial_\psi I_\uparrow + \frac{\sqrt{3}qea}{2r^3 \sqrt{h(r)}} + \tilde{m} \right) = 0,
\end{aligned} \tag{53}$$

$$\frac{2B}{r\sqrt{k(r)}} \left(-\partial_\theta I_\uparrow - \frac{i}{\sin \theta} \partial_\phi I_\uparrow + i \cot \theta \partial_\psi I_\uparrow \right) = 0, \tag{54}$$

$$\begin{aligned}
& B \left(\sqrt{\frac{h(r)}{w(r)}} \left[-\beta \partial_{\bar{t}} I_\uparrow + f(r) \partial_\psi I_\uparrow + \frac{\sqrt{3}qe}{2r^2} \left(1 + \frac{af(r)}{2} \right) \right] - \right. \\
& \quad \left. \frac{\sqrt{w(r)}}{k(r)} \partial_r I_\uparrow \right) + A \left(\frac{2}{r\sqrt{h(r)}} \partial_\psi I_\uparrow + \frac{\sqrt{3}qea}{2r^3 \sqrt{h(r)}} + \tilde{m} \right) = 0,
\end{aligned} \tag{55}$$

$$\frac{2A}{r\sqrt{k(r)}} \left(-\partial_\theta I_\uparrow - \frac{i}{\sin \theta} \partial_\phi I_\uparrow + i \cot \theta \partial_\psi I_\uparrow \right) = 0. \tag{56}$$

It follows immediately from the obtained equations that all the variables can be separated. The action I_\uparrow is supposed to take form:

$$I_\uparrow = -Et + J\phi + L\psi + R(r) + \Theta(\theta), \tag{57}$$

where E is the energy of emitted Dirac particles and J and L are angular momenta corresponding to the angles ϕ and ψ respectively. We also remark that in case of scalar particles we initially

assumed that just part of variables can be separated and after little algebra we concluded that we had complete separation of variables. The complete separation of variables for Dirac particles follows from the written equations (53)-(56). Having inserted the ansatz (57) into equations (53)-(56) we obtain:

$$B \left(\sqrt{\frac{h(r)}{w(r)}} \left[\beta E + f(r)L + \frac{\sqrt{3}qe}{2r^2} \left(1 + \frac{af(r)}{2} \right) \right] - \frac{\sqrt{w(r)}}{k(r)} R'(r) \right) + A \left(\frac{1}{r\sqrt{h(r)}} \left(2L + \frac{\sqrt{3}qea}{2r^2} \right) + \tilde{m} \right) = 0 \quad (58)$$

$$-\frac{2B}{r\sqrt{k(r)}} \left(\Theta' + \frac{iJ}{\sin \theta} - iL \cot \theta \right) = 0 \quad (59)$$

$$A \left(-\sqrt{\frac{h(r)}{w(r)}} \left[\beta E + f(r)L + \frac{\sqrt{3}qe}{2r^2} \left(1 + \frac{af(r)}{2} \right) \right] - \frac{\sqrt{w(r)}}{k(r)} R'(r) \right) + A \left(\frac{-1}{r\sqrt{h(r)}} \left(2L + \frac{\sqrt{3}qea}{2r^2} \right) + \tilde{m} \right) = 0 \quad (60)$$

$$-\frac{2A}{r\sqrt{k(r)}} \left(\Theta' + \frac{iJ}{\sin \theta} - iL \cot \theta \right) = 0 \quad (61)$$

System of equations (58) and (60) has a nontrivial solution for parameters A and B if and only if the determinant of corresponding matrix is equal to zero. That requirement gives rise to the following equation:

$$R'(r) = \frac{\beta k(r)\sqrt{h(r)}}{w(r)} \left[\left(E + \frac{f(r)}{\beta} L + \frac{\sqrt{3}qe}{2\beta r^2} \left(1 + \frac{af(r)}{2} \right) \right)^2 - \frac{w(r)}{\beta^2 h(r)} \left(\frac{1}{r^2 h(r)} \left(2L + \frac{\sqrt{3}qea}{2r^2} \right)^2 - \tilde{m}^2 \right) \right]^{1/2} \quad (62)$$

The structure of the obtained relation is similar to the relation for scalar particles (24) and it is not accidental coincidence because we consider quasi-classical approximation for both types of particles. In the vicinity of the horizon point we again use the decomposition $w(r) = w'(r_+)(r - r_+) = 2(r_+^2 - r_-^2)(r - r_+)/r_+^3$. So we can write:

$$R'(r) = \frac{\beta r_\infty^2 - r_-^2}{2 r_\infty^2 - r_+^2} \frac{\sqrt{r_+^6 - a^2 q^2 - 2a^2(m+q)r_+^2}}{(r_+^2 - r_-^2)(r - r_+)} \left(E + \frac{f(r_+)L}{\beta} + \frac{\sqrt{3}qe}{2\beta r_+^2} \left(1 + \frac{af(r_+)}{2} \right) \right) \quad (63)$$

To obtain temperature for Dirac particles we follow the same steps that we have made for scalar particles. The similarity of the obtained relations for the derivatives of radial part of the action brings us to the conclusion that temperature for tunnelling fermions has to be the same as for scalar particles and will be defined by the expression (31). We also note that equality of temperatures for scalar particles and Dirac fermions follows from the fact that both types of particles are examined quasi-classically.

Now we proceed to the tunnelling of Dirac particles in case of black hole's metric (32). We note that fermion tunnelling from the squashed black hole in Gödel universe was investigated in [43] but the author considered just particular case of nonrotating black hole without charge. To write the Dirac equation gamma matrices should be defined. The gamma matrices take the form:

$$\begin{aligned}\hat{\gamma}^t &= \frac{2}{r} \sqrt{\frac{h(r)}{V(r)}} \hat{\gamma}^0, \quad \hat{\gamma}^r = \frac{\sqrt{V(r)}}{\chi(r)} \hat{\gamma}^3, \quad \hat{\gamma}^\theta = \frac{2}{r\sqrt{\chi(r)}} \hat{\gamma}^1, \\ \hat{\gamma}^\phi &= \frac{2}{r\sqrt{\chi(r)} \sin \theta} \hat{\gamma}^2, \quad \hat{\gamma}^\psi = \frac{2g(r)}{r\sqrt{h(r)V(r)}} \hat{\gamma}^0 - \frac{2 \cot \theta}{r\sqrt{k(r)}} \hat{\gamma}^2 + \frac{1}{r\sqrt{h(r)}} \hat{\gamma}^4\end{aligned}\tag{64}$$

The Dirac equation can be written as follows:

$$\begin{aligned}& i \left(\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}} \hat{\gamma}^0 \partial_t + \frac{\sqrt{V(r)}}{\chi(r)} \hat{\gamma}^3 \partial_r + \frac{2}{r\sqrt{\chi(r)}} \hat{\gamma}^1 \partial_\theta + \frac{2}{r\sqrt{\chi(r)} \sin \theta} \hat{\gamma}^2 \partial_\phi + \right. \\ & \left. \left[\frac{2g(r)}{r\sqrt{h(r)V(r)}} \hat{\gamma}^0 - \frac{2 \cot \theta}{r\sqrt{k(r)}} \hat{\gamma}^2 + \frac{1}{\sqrt{h(r)}} \hat{\gamma}^4 \right] \partial_\psi \right) \Psi + \frac{e}{\hbar} \left(\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}} \hat{\gamma}^0 \times \right. \\ & \left. \left(\frac{\sqrt{3}q}{2r^2} - \Phi \right) + \frac{\sqrt{3} \cos \theta}{r\sqrt{\chi(r)} \sin \theta} \hat{\gamma}^2 \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) + \left[\frac{2g(r)}{r\sqrt{h(r)V(r)}} \hat{\gamma}^0 - \right. \right. \\ & \left. \left. \frac{2 \cot \theta}{r\sqrt{k(r)}} \hat{\gamma}^2 + \frac{1}{\sqrt{h(r)}} \hat{\gamma}^4 \right] \frac{\sqrt{3}}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right) \Psi + \frac{m}{\hbar} \Psi = 0\end{aligned}\tag{65}$$

Here we also omitted spin connection terms.

Now we suppose that the wavefunction takes the same form as in the previous case: (52). Having substituted the wavefunction (52) into the equation (65) we arrive at:

$$\begin{aligned}& B \left[-\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}} \partial_t I_\uparrow - \frac{\sqrt{V(r)}}{\chi(r)} \partial_r I_\uparrow - \frac{2g(r)}{r\sqrt{h(r)V(r)}} \partial_\psi I_\uparrow + \right. \\ & \left. \frac{2e\sqrt{h(r)}}{r\sqrt{V(r)}} \left(\frac{\sqrt{3}q}{2r^2} - \Phi \right) + \frac{2eg(r)}{r\sqrt{h(r)V(r)}} \frac{\sqrt{3}}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right] + \\ & A \left[\frac{1}{\sqrt{h(r)}} \partial_\psi I_\uparrow - \frac{1}{\sqrt{h(r)}} \frac{\sqrt{3}}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) + \tilde{m} \right] = 0\end{aligned}\tag{66}$$

$$\frac{2B}{r\sqrt{\chi(r)}} \left[-\partial_\theta I_\uparrow - \frac{i}{\sin \theta} \partial_\phi I_\uparrow - i \cot \theta \partial_\psi I_\uparrow \right] = 0\tag{67}$$

$$\begin{aligned}& A \left[\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}} \partial_t I_\uparrow - \frac{\sqrt{V(r)}}{\chi(r)} \partial_r I_\uparrow + \frac{2g(r)}{r\sqrt{h(r)V(r)}} \partial_\psi I_\uparrow - \right. \\ & \left. \frac{2e\sqrt{h(r)}}{r\sqrt{V(r)}} \left(\frac{\sqrt{3}q}{2r^2} - \Phi \right) - \frac{\sqrt{3}eg(r)}{r\sqrt{h(r)V(r)}} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right] + \\ & B \left[-\frac{1}{\sqrt{h(r)}} \partial_\psi I_\uparrow + \frac{1}{\sqrt{h(r)}} \frac{\sqrt{3}}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) + \tilde{m} \right] = 0\end{aligned}\tag{68}$$

$$\frac{2A}{r\sqrt{\chi(r)}} \left[-\partial_\theta I_\uparrow - \frac{i}{\sin \theta} \partial_\phi I_\uparrow - i \cot \theta \partial_\psi I_\uparrow \right] = 0\tag{69}$$

The equations (67) and (69) show that angular variables can be separated from the radial one. So the action takes the form (57) again. Then the pair of equations (66) and (68) give rise to the following system:

$$\begin{aligned} & B \left[\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}} E - \frac{\sqrt{V(r)}}{\chi(r)} R'(r) - \frac{2g(r)}{r\sqrt{h(r)V(r)}} L + \right. \\ & \left. \frac{2e\sqrt{h(r)}}{r\sqrt{V(r)}} \left(\frac{\sqrt{3}q}{2r^2} - \Phi \right) + \frac{2eg(r)}{r\sqrt{h(r)V(r)}} \frac{\sqrt{3}}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right] + \\ & A \left[\frac{1}{\sqrt{h(r)}} \left(L - \frac{\sqrt{3}}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right) + \tilde{m} \right] = 0 \end{aligned} \quad (70)$$

$$\begin{aligned} & A \left[-\frac{2\sqrt{h(r)}}{r\sqrt{V(r)}} E - \frac{\sqrt{V(r)}}{\chi(r)} R'(r) + \frac{2g(r)}{r\sqrt{h(r)V(r)}} L - \right. \\ & \left. \frac{2e\sqrt{h(r)}}{r\sqrt{V(r)}} \left(\frac{\sqrt{3}q}{2r^2} - \Phi \right) - \frac{\sqrt{3}eg(r)}{r\sqrt{h(r)V(r)}} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right] + \\ & B \left[-\frac{1}{\sqrt{h(r)}} \left(L - \frac{\sqrt{3}}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right) + \tilde{m} \right] = 0 \end{aligned} \quad (71)$$

The equations (67) and (69) are identical and lead to equation for the angular part of the action. This equation is the same as for the previous case (59) and (61):

$$\Theta'(\theta) + \frac{iJ}{\sin\theta} - iL \cot\theta = 0 \quad (72)$$

So the angular part of the action takes the same form as for the metric (4).

In order to get equation for the derivative of the radial part of the action we make use the same arguments as in previous case. So we arrive at the equation:

$$\begin{aligned} R'(r) = & \frac{2\chi(r)\sqrt{h(r)}}{rV(r)} \left[\left(E + e \left(\frac{\sqrt{3}q}{2r^2} - \Phi \right) - \frac{g(r)}{h(r)} \left[L - \frac{e\sqrt{3}}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right] \right)^2 \right. \\ & \left. + \frac{r^2V(r)}{4h(r)} \left(\tilde{m}^2 - \frac{1}{h(r)} \left[L + \frac{\sqrt{3}}{2} \left(jr^2 + 2jq - \frac{aq}{2r^2} \right) \right]^2 \right) \right]^{1/2} \end{aligned} \quad (73)$$

Having integrated the latter equation in the vicinity of horizon point and taking the imaginary part we obtain the relation (43). So we conclude that temperature for tunnelling Dirac fermions in case of Gödel universe will be the same as for scalar particles and takes form (45).

4 Conclusions

We have investigated tunnelling of scalar particles and Dirac fermions from the squashed charged rotating black holes in five dimensional case (4). The same procedure has been accomplished for similar type of black hole but in Gödel universe (32). To consider tunnelling of scalar particles we have made use of the Hamilton-Jacobi approach which is based on the examination of the Hamilton-Jacobi equation (3). As we noted earlier the Hamilton-Jacobi equation we used is the quasi-classical limit of the Klein-Gordon equation (1) that describes scalar particles in quantum

mechanics. To find temperature of a black hole the imaginary part of the action which satisfies the Hamilton-Jacobi equation should be found. This fact that the action takes complex values is the direct consequence of the tunnelling process thorough the horizon. From the point of view of mathematics complex values for the action appear due to the integration on the interval which includes a pole of integrand (the horizon point is the simple pole for the integrand). The imaginary part of the action allows us to obtain the Boltzmann factor when we calculate tunnelling probability (29). The expressions for temperature of the black hole that we have obtained here take the same form as it was obtained by other method [31, 32].

To consider tunnelling of Dirac particles the approach proposed by Kerner and Mann [11] has been employed. In case of fermions we also restrict oneself to the quasi-classical approximation. For this purpose we have chosen the wave function of Dirac equation in the form (52) and supposed that coefficients $A(t, r, \theta, \phi, \psi)$ and $B(t, r, \theta, \phi, \psi)$ are constant because we take into consideration only the terms proportional to \hbar^{-1} . From the written system of equations it follows immediately that variables can be separated. Then similarly to the case of scalar particles we have singled out the equation for the derivative of radial part of the action I_{\uparrow} . We note that obtained expressions for the radial part of the action in the vicinity of horizon for Dirac fermions are identical to corresponding expressions for scalar particles. It should be noticed that angular part of the action which corresponds to the angle θ can take complex values. But in comparison to the radial part where imaginary parts for outgoing and ingoing particles take opposite sign for the angular part $\Theta(\theta)$ they take the same sign and can be cancelled out. So the angular part of the action does not influence on the determination of temperature. As a consequence, temperature we have found for tunnelling fermions is the same as for scalar particles.

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